

First-Order Special Relativistic Corrections to Kepler's Orbits

Tyler J. Lemmon and Antonio R. Mondragon

Physics Department, Colorado College, Colorado Springs, Colorado 80903

(Dated: December 30, 2010)

Abstract

Beginning with a Lagrangian that is consistent with both Newtonian gravity and the momentum-velocity relation of special relativity, an approximate relativistic orbit equation is derived that describes relativistic corrections to Keplerian orbits. Specifically, corrections to a Keplerian orbit due to special relativity include: precession of perihelion, reduced radius of circular orbit, and increased eccentricity. The prediction for the rate of precession of perihelion of Mercury is in agreement with existing calculations using only special relativity, and is one sixth that derived from general relativity. All three of these corrections are qualitatively correct, though suppressed when compared to the more accurate general-relativistic corrections in this limit. The resulting orbit equation has the same form as that derived from general relativity and is easily compared to that describing Kepler's orbits. This treatment of the relativistic central-mass problem is complementary to other solutions to the relativistic Kepler problem, and is approachable by undergraduate physics majors whom have not had a course dedicated to relativity.

PACS numbers: 04.25.Nx, 95.30.Sf, 03.30.+p, 95.10.Ce

I. INTRODUCTION

The relativistic contribution to the rate of precession of perihelion of Mercury is calculated accurately using general relativity [1–6]. However, the problem is commonly discussed in undergraduate and graduate classical mechanics textbooks, without introduction of an entirely new, metric theory of gravity. One approach [7–11] is to define a Lagrangian that is consistent with both Newtonian gravity and the momentum-velocity relation of special relativity. The resulting equation of motion is solved perturbatively, and an approximate rate of precession of perihelion of Mercury is extracted. This approach is satisfying in that all steps are proved, and a brief introduction to special relativity is included. On the other hand, one must be content with an approximate rate of precession that is about one-sixth the correct value. Another approach [12–16] is that of a mathematical exercise and history lesson. A modification to Newtonian gravity is given, without proof, resulting in an equation of motion that is the same as that derived from general relativity. The equation of motion is then solved using appropriate approximations, and the correct rate of precession of perihelion of Mercury is extracted. Both approaches provide an opportunity for students of classical mechanics to understand that relativity is responsible for a small contribution to perihelic precession and to calculate that contribution.

We present a review of the approach using only special relativity, followed by an alternative solution of the equation of motion derived from Lagrange’s equations. An approximate rate of perihelic precession is derived that agrees with established calculations. This effect arises as one of several small corrections to Kepler’s orbits, including reduced radius of circular orbit and increased eccentricity. The method of solution makes use of coordinate transformations and the correspondence principle, rather than the standard perturbative techniques, and is approachable by undergraduate physics majors.

A relativistic particle of mass m orbiting a central mass M is commonly described by the Lagrangian [7–11, 17–22]

$$L = -mc^2\gamma^{-1} - U(r), \quad (1)$$

where: $\gamma^{-1} \equiv \sqrt{1 - v^2/c^2}$; $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$; and $U(r) = -GMm/r$. (G is Newton’s universal gravitational constant, and c is the speed of light in vacuum.) The equations of motion follow from Lagrange’s equations,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad (2)$$

for each of $\{q_i\} = \{\theta, r\}$, where $\dot{q}_i \equiv dq_i/dt$. The results are:

$$\frac{d}{dt}[\gamma r^2 \dot{\theta}] = 0, \quad (3)$$

which implies that $\ell \equiv \gamma r^2 \dot{\theta} = \text{constant}$; and

$$\gamma \ddot{r} + \dot{\gamma} \dot{r} + \frac{GM}{r^2} - \gamma r \dot{\theta}^2 = 0. \quad (4)$$

The first of these [Eq. (3)] is the relativistic analogue to the Newtonian equation for the conservation of angular momentum per unit mass, and is used to eliminate $\dot{\theta}$ in Eq. (4),

$$\gamma r \dot{\theta}^2 = \frac{\ell^2}{\gamma r^3}. \quad (5)$$

Time is eliminated by successive applications of the chain rule, together with the conserved angular momentum [23–27]:

$$\dot{r} = -\frac{\ell}{\gamma} \frac{d}{d\theta} \frac{1}{r}; \quad (6)$$

and, therefore,

$$\gamma \ddot{r} = -\dot{\gamma} \dot{r} - \frac{\ell^2}{\gamma r^2} \frac{d^2}{d\theta^2} \frac{1}{r}. \quad (7)$$

Substituting Eqs. (5) and (7) into the equation of motion Eq. (4) results in

$$\ell^2 \frac{d^2}{d\theta^2} \frac{1}{r} - GM\gamma + \frac{\ell^2}{r} = 0. \quad (8)$$

We anticipate a solution of Eq. (8) that is near Keplerian and introduce the radius of a circular orbit for a nonrelativistic particle with the same angular momentum, $r_c \equiv \ell^2/GM$. The result is

$$\frac{d^2}{d\theta^2} \frac{r_c}{r} + \frac{r_c}{r} = 1 + \lambda, \quad (9)$$

where $\lambda \equiv \gamma - 1$ is a velocity-dependent correction to Newtonian orbits due to special relativity. The conic-sections of Newtonian mechanics [28–33] are recovered by setting $\lambda = 0$ ($c \rightarrow \infty$):

$$\frac{d^2}{d\theta^2} \frac{r_c}{r} + \frac{r_c}{r} = 1, \quad (10)$$

which implies that

$$\frac{r_c}{r} = 1 + e \cos \theta, \quad (11)$$

where e is the eccentricity.

II. KEPLERIAN LIMIT AND ORBIT EQUATION

The planets of our solar system are described by near-circular orbits ($e \ll 1$) and require only small relativistic corrections ($v/c \ll 1$). Mercury has the largest eccentricity ($e \approx 0.2$), and the next largest is that of Mars ($e \approx 0.09$). Therefore, λ [defined after Eq. (9)] is taken to be a small relativistic correction to near-circular orbits of Newtonian mechanics (Keplerian orbits). Neglecting the radial component of the velocity, which is small compared to the tangential component, and expanding γ to first order in v^2/c^2 results in

$$\lambda \approx \frac{1}{2}(r\dot{\theta}/c)^2. \quad (12)$$

(See Sec. IV and App. A for a thorough discussion of this approximation.) Once again using the angular momentum to eliminate $\dot{\theta}$, this may be expressed as $\lambda \approx \frac{1}{2}(\ell/rc)^2\gamma^{-2} \approx \frac{1}{2}(\ell/rc)^2(1-2\lambda)$, or

$$\lambda \approx \frac{1}{2}(\ell/rc)^2. \quad (13)$$

The equation of motion Eq. (9) is now expressed approximately as

$$\frac{d^2}{d\theta^2} \frac{r_c}{r} + \frac{r_c}{r} \approx 1 + \frac{1}{2}\epsilon \left(\frac{r_c}{r}\right)^2. \quad (14)$$

where $\epsilon \equiv (GM/\ell c)^2$. The conic-sections of Newtonian mechanics, Eq. (10) and Eq. (11), are now recovered by setting $\epsilon = 0$ ($c \rightarrow \infty$). The solution of Eq. (14) for $\epsilon \neq 0$ approximately describes Keplerian orbits with small corrections due to special relativity.

If ϵ is taken to be a small relativistic correction to Keplerian orbits, it is convenient to make the change of variable $1/s \equiv r_c/r - 1 \ll 1$. The last term on the right-hand-side of Eq. (14) is then approximated as $(r_c/r)^2 \approx 1 + 2/s$, resulting in a linear differential equation for $1/s(\theta)$:

$$\frac{2}{\epsilon} \frac{d^2}{d\theta^2} \frac{1}{s} + \frac{2(1-\epsilon)}{\epsilon} \frac{1}{s} \approx 1. \quad (15)$$

The additional change of variable $\alpha \equiv \theta\sqrt{1-\epsilon}$ results in the familiar form:

$$\frac{d^2}{d\alpha^2} \frac{s_c}{s} + \frac{s_c}{s} \approx 1, \quad (16)$$

where $s_c \equiv 2(1-\epsilon)/\epsilon$. The solution is similar to that of Eq. (10):

$$\frac{s_c}{s} \approx 1 + A \cos \alpha, \quad (17)$$

where A is an arbitrary constant of integration. In terms of the original coordinates, Eq. (17) becomes

$$\frac{\tilde{r}_c}{r} \approx 1 + \tilde{e} \cos \tilde{\kappa} \theta, \quad (18)$$

where

$$\tilde{r}_c \equiv r_c \frac{1 - \epsilon}{1 - \frac{1}{2}\epsilon}, \quad (19)$$

$$\tilde{e} \equiv \frac{\frac{1}{2}\epsilon A}{1 - \frac{1}{2}\epsilon}, \quad (20)$$

$$\tilde{\kappa} \equiv (1 - \epsilon)^{\frac{1}{2}}. \quad (21)$$

According to the correspondence principle, Kepler's orbits, Eq. (11), must be recovered in the limit $\epsilon \rightarrow 0$ ($c \rightarrow \infty$), so that $\frac{1}{2}\epsilon A \equiv e$ is the eccentricity of Newtonian mechanics. To first order in ϵ , Eqs. (19)–(21) become

$$\tilde{r}_c \approx r_c(1 - \frac{1}{2}\epsilon), \quad (22)$$

$$\tilde{e} \approx e(1 + \frac{1}{2}\epsilon), \quad (23)$$

$$\tilde{\kappa} \approx 1 - \frac{1}{2}\epsilon, \quad (24)$$

so that relativistic orbits in this limit are described concisely by

$$\frac{r_c(1 - \frac{1}{2}\epsilon)}{r} \approx 1 + e(1 + \frac{1}{2}\epsilon) \cos(1 - \frac{1}{2}\epsilon)\theta. \quad (25)$$

This approximate orbit equation has the same form as that derived from general relativity [34],

$$\frac{r_c(1 - 3\epsilon)}{r} \approx 1 + e(1 + 3\epsilon) \cos(1 - 3\epsilon)\theta, \quad (26)$$

and clearly displays three characteristics: precession of perihelion; reduced radius of circular orbit; and increased eccentricity.

III. CHARACTERISTICS OF NEAR-KEPLERIAN ORBITS

The approximate orbit equation Eq. (25) predicts a shift in the perihelion through an angle

$$\Delta\theta \equiv 2\pi(\tilde{\kappa}^{-1} - 1) \approx \pi\epsilon \quad (27)$$

per revolution. This prediction is identical to that derived using the standard approach [7–11, 35] to incorporating special relativity into the Kepler problem, and is compared to observations assuming

that the relativistic and Keplerian angular momenta are approximately equal. For a Keplerian orbit [28–33] $\ell^2 = GMa(1 - e^2)$, where $G = 6.670 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$, $M = 1.989 \times 10^{30} \text{ kg}$ is the mass of the Sun, and a and e are the semi-major axis and eccentricity of the orbit, respectively. Therefore, the relativistic correction defined after Eq. (14),

$$\epsilon \approx \frac{GM}{c^2 a(1 - e^2)}, \quad (28)$$

is largest for planets closest to the Sun and for planets with very eccentric orbits. For Mercury [36, 37] $a = 5.79 \times 10^{10} \text{ m}$ and $e = 0.2056$, so that $\epsilon \approx 2.66 \times 10^{-8}$. (The speed of light is taken to be $c^2 = 8.987554 \times 10^{16} \text{ m}^2/\text{s}^2$.) According to Eq. (27), Mercury precesses through an angle

$$\Delta\theta \approx \frac{\pi GM}{c^2 a(1 - e^2)} = 8.36 \times 10^{-8} \text{ rad} \quad (29)$$

per revolution. This angle is very small and is usually expressed cumulatively in arc seconds per century. The orbital period of Mercury is 0.24085 terrestrial years, so that

$$\Delta\Theta \equiv \frac{100 \text{ yr}}{0.24085 \text{ yr}} \times \frac{360 \times 60 \times 60}{2\pi} \times \Delta\theta \quad (30a)$$

$$\approx 7.16 \text{ arcsec/century}. \quad (30b)$$

The general relativistic (GR) treatment results in a prediction of 43.0 arcsec/century [34–55], and agrees with the observed precession of perihelia of the inner planets [37–46, 56–59]. Historically, this contribution to the precession of perihelion of Mercury’s orbit precisely accounted for the observed discrepancy, serving as the first triumph of the general theory of relativity [1–4]. The present approach, using only special relativity, accounts for about one-sixth of the observed discrepancy, Eq. (30b). Precession due to special relativity is illustrated in Fig. 1.

The approximate relativistic orbit equation Eq. (25) [or Eq. (18) together with Eqs. (22)–(24)] predicts that a relativistic orbit in this limit has a reduced radius of circular orbit ($e = 0$). This characteristic is not discussed in the standard approach to incorporating special relativity into the Kepler problem, but is consistent with the GR description. An effective potential naturally arises in the GR treatment of the central-mass problem [34, 36, 40, 41, 43, 46, 53, 54, 60, 61],

$$V_{\text{eff}} \equiv -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{c^2 r^3}, \quad (31)$$

that reduces to the Newtonian effective potential in the limit $c \rightarrow \infty$. In the Keplerian limit, the GR angular momentum per unit mass ℓ is also taken to be approximately equal to that for a

Keplerian orbit [34, 36–43, 46, 54, 61, 62]. Minimizing V_{eff} with respect to r results in the radius of a stable circular orbit,

$$R_c = \frac{1}{2}r_c + \frac{1}{2}r_c\sqrt{1 - 12\epsilon} \approx r_c(1 - 3\epsilon), \quad (32)$$

so that the radius of circular orbit is predicted to be reduced: $R_c - r_c \approx -3\epsilon r_c$. (There is also an unstable circular orbit. See Fig. 2.) This reduction in radius of circular orbit is six times that predicted by the present treatment using only special relativity, for which $\tilde{r}_c - r_c \approx -\frac{1}{2}\epsilon r_c$. [See Eq. (22).]

Most discussions of the GR effective potential Eq. (31) emphasize relativistic capture, rather than reduced radius of circular orbit. The $1/r^3$ term in Eq. (31) contributes negatively to the effective potential, resulting in a finite centrifugal barrier and affecting orbits very near the central mass (large-velocity orbits). (See Fig. 2.) This purely GR effect is not expected to be described by the approximate orbit equation Eq. (25), which is derived using only special relativity and assumes orbits very far from the central mass (small-velocity orbits).

An additional characteristic of relativistic orbits is that of increased eccentricity. Equation (25) predicts that a relativistic orbit will have increased eccentricity, when compared to a Keplerian orbit with the same angular momentum: $\tilde{e} - e \approx \frac{1}{2}\epsilon e$. [See Eq. 23.] This characteristic of relativistic orbits also is not discussed in the standard approach to incorporating special relativity into the Kepler problem, but is consistent with the GR description. The GR orbit equation in this Keplerian limit Eq. (26) predicts an increase in eccentricity $\tilde{e} - e \approx 3\epsilon e$, which is six times that predicted by the present treatment using only special relativity.

IV. DISCUSSION

The approximate orbit equation in Eq. (25) provides small corrections to Kepler's orbits due to special relativity. [Compare Eqs. (25) and (11).] A systematic verification may be carried out by substituting Eq. (25) into Eq. (14), keeping terms of orders e , ϵ , and $e\epsilon$ only. The domain of validity is expressed by subjecting the solution Eq. (25) to the condition

$$\frac{r_c}{r} - 1 \ll 1 \quad (33)$$

for the smallest value of r . Evaluating the orbit equation Eq. (25) at the perihelion (r_p) results in

$$\frac{r_c}{r_p} = \frac{1 + e(1 + \frac{1}{2}\epsilon)}{1 - \frac{1}{2}\epsilon}. \quad (34)$$

The substitution of this result into Eq. (33) results in the domain of validity:

$$e(1 + \frac{1}{2}\epsilon) + \epsilon \ll 1. \quad (35)$$

Therefore, the relativistic eccentricity $\tilde{e} = e(1 + \frac{1}{2}\epsilon) \ll 1$, and Eq. (25) is limited to describing relativistic corrections to near-circular (Keplerian) orbits. Also, the relativistic correction $\epsilon \ll 1$, and thus the orbit equation Eq. (25) is valid only for small relativistic corrections.

In Sec. II the relativistic correction to Keplerian orbits λ [defined after Eq. (9)] is approximated by: neglecting the radial component of the velocity, $v^2 = \dot{r}^2 + (r\dot{\theta})^2 \approx (r\dot{\theta})^2$; and keeping terms only up to first order in the expansion $\gamma \approx 1 + \frac{1}{2}(v/c)^2$. Neglecting the radial component of the velocity in the relativistic correction λ is consistent with the assumption of near-circular, approximately Keplerian orbits, and is complementary to the assumption $r_c/r - 1 \ll 1$ preceding Eq. (15). It is emphasized that the radial component of the velocity is neglected only in the relativistic correction λ ; it is not neglected in the derivation of the relativistic equation of motion Eq. (9). That there is no explicit appearance of \dot{r} in the relativistic equation of motion Eq. (9), other than in the definition of γ , is due to a fortunate cancellation after Eq. (7). Furthermore, the approximate orbit equation Eq. (25) has the same form as that [Eq. (26)] arising from the GR description, in which the radial component of the velocity is not explicitly neglected.

A first-order series approximation for γ is consistent with the Keplerian limit. Interestingly, however, the problem is soluble without truncating the series. (See Appendix A.) This more elaborate derivation yields diminishing returns due to the complementary assumption $r_c/r - 1 \ll 1$, which implicitly constrains the velocity to be much smaller than the speed of light. The resulting approximate orbit equation [Eq. (A13)],

$$\frac{r_c(1 - \frac{1}{2}\epsilon)}{r} \approx 1 + e(1 + \frac{5}{4}\epsilon) \cos(1 - \frac{1}{2}\epsilon)\theta, \quad (36)$$

is almost identical to that derived using a much simpler approach in Sec. II [Eq. (25)]. Although this alternative orbit equation predicts a relativistic correction to eccentricity that is nearly one-half that of the GR result, it lacks the symmetry of the GR orbit equation Eq. (26), and it does not provide significant further qualitative understanding of relativistic corrections to Keplerian orbits.

V. CONCLUSION

The present approach to incorporating special relativity into the Kepler problem results in an approximate orbit equation [Eq. (25)] that has the same form as that derived from general rela-

tivity in this limit [Eq. (26)] and is easily compared to that describing Kepler's orbits [Eq. (10)]. This orbit equation clearly describes three corrections to a Keplerian orbit due to special relativity: precession of perihelion; reduced radius of circular orbit; and increased eccentricity. The predicted rate of precession of perihelion of Mercury is identical to established calculations using only special relativity. Each of these corrections is exactly one-sixth of the corresponding correction described by general relativity in the Keplerian limit.

This derivation of an approximate orbit equation is complementary to existing calculations of the rate of precession of perihelion of Mercury using only special relativity. The central-mass problem is described by a Lagrangian that is consistent with both Newtonian gravity and the momentum-velocity relation of special relativity. However, coordinate transformations and the correspondence principle are used to solve the equations of motion resulting from Lagrange's equations, rather than the standard perturbative methods. The resulting closed-form, approximate orbit equation exhibits several characteristics of relativistic orbits at once, but is limited to describing small relativistic corrections to approximately Newtonian, near-circular orbits. This orbit equation, derived using only special relativity, provides a qualitative description of corrections to Keplerian orbits due to general relativity. Exact solutions to the special relativistic Kepler problem require a thorough understanding of special relativistic mechanics [63, 64] and are, therefore, inaccessible to most undergraduate physics majors. The present approach and method of solution is understandable to nonspecialists, including undergraduate physics majors whom have not had a course dedicated to relativity.

Appendix A

For near-circular orbits the radial component of the velocity is small compared to the tangential component and is neglected in the relativistic correction $\lambda = \gamma - 1$. [See Eq. (9).] An exact infinite-series representation,

$$\gamma \approx \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \left(\frac{r\dot{\theta}}{c} \right)^{2n}, \quad (\text{A1})$$

is used, rather than the approximate first-order truncated series used in Sec. II. (The remainder of this derivation parallels that of Sec. II.) Conservation of angular momentum is used to eliminate $\dot{\theta}$, resulting in

$$\lambda \approx \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2} \left(\frac{\ell}{cr_c} \right)^{2n} \left(\frac{r_c}{r} \right)^{2n} \right] - 1, \quad (\text{A2})$$

in terms of which the equation of motion Eq. (9) is now expressed approximately as

$$\frac{d^2}{d\theta^2} \frac{r_c}{r} + \frac{r_c}{r} \approx 1 + \lambda. \quad (\text{A3})$$

The conic-sections of Newtonian mechanics, Eqs. (10) and (11), are recovered by setting $c \rightarrow \infty$.

For near-circular, approximately Keplerian orbits it is convenient to make the change of variable $1/s \equiv r_c/r - 1 \ll 1$, so that $(r_c/r)^{2n} \approx 1 + 2n/s$. In terms of this new variable, the relativistic correction Eq. (A2) is

$$\lambda \approx (\gamma_\epsilon - 1) + \epsilon \gamma_\epsilon^3 \frac{1}{s}, \quad (\text{A4})$$

where $\gamma_\epsilon \equiv 1/\sqrt{1-\epsilon}$, and the following series is identified:

$$\sum_{n=0}^{\infty} \frac{2n(2n)!}{2^{2n}(n!)^2} \left(\frac{\ell}{cr_c} \right)^{2n} = \epsilon \gamma_\epsilon^3. \quad (\text{A5})$$

Thus, the equation of motion Eq. (A3) is linearized:

$$\frac{1}{\gamma_\epsilon - 1} \frac{d^2}{d\theta^2} \frac{1}{s} + \frac{1 - \epsilon \gamma_\epsilon^3}{\gamma_\epsilon - 1} \frac{1}{s} \approx 1. \quad (\text{A6})$$

The additional change of variable $\alpha \equiv \theta \sqrt{1 - \epsilon \gamma_\epsilon^3}$ results in the familiar form:

$$\frac{d^2}{d\alpha^2} \frac{s_c}{s} + \frac{s_c}{s} \approx 1, \quad (\text{A7})$$

where $s_c \equiv (1 - \epsilon \gamma_\epsilon^3)/(\gamma_\epsilon - 1)$. The solution is similar to that of Eq. (10):

$$\frac{s_c}{s} \approx 1 + A \cos \alpha, \quad (\text{A8})$$

where A is an arbitrary constant of integration. In terms of the original coordinates, Eq. (A8) becomes

$$\frac{\tilde{r}_c}{r} \approx 1 + \tilde{e} \cos \tilde{\kappa} \theta, \quad (\text{A9})$$

where (including first-order approximations)

$$\tilde{r}_c \equiv r_c \frac{\gamma_\epsilon^{-1}(1-\epsilon) - \epsilon}{1-2\epsilon} \approx r_c(1 - \frac{1}{2}\epsilon), \quad (\text{A10})$$

$$\tilde{e} \equiv A \frac{(1 - \gamma_\epsilon^{-1})(1 - \epsilon)}{1 - 2\epsilon} \approx \frac{1}{2}\epsilon A(1 + \frac{5}{4}\epsilon), \quad (\text{A11})$$

$$\tilde{\kappa} \equiv (1 - \epsilon \gamma_\epsilon^3)^{\frac{1}{2}} \approx 1 - \frac{1}{2}\epsilon. \quad (\text{A12})$$

According to the correspondence principle, Kepler's orbits [Eq. (11) with $e < 1$] must be recovered in the limit $\epsilon \rightarrow 0$ ($c \rightarrow \infty$), so that $\frac{1}{2}\epsilon A \equiv e$ is the eccentricity of Newtonian mechanics. Therefore, relativistic orbits in this limit are described concisely by

$$\frac{r_c(1 - \frac{1}{2}\epsilon)}{r} \approx 1 + e(1 + \frac{5}{4}\epsilon) \cos(1 - \frac{1}{2}\epsilon)\theta. \quad (\text{A13})$$

This alternative orbit equation differs from that [Eq. (25)] derived using the much simpler approach in Sec. II only in the relativistic correction to eccentricity. See the discussion at the end of Sec. IV.

-
- [1] A. Einstein, "Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie," Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1915**, 831–839 (1915).
 - [2] A. Einstein, "Explanation of the perihelion motion of mercury from the general theory of relativity," in *The Collected Papers of Albert Einstein*, translated by A. Engel (Princeton University Press, Princeton, 1997), Vol. 6, pp. 112–116. This article is the English translation of Ref. 1.
 - [3] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie; von A. Einstein," Ann. d. Phys. **354** (7), 769–822 (1916).
 - [4] Reference 2, pp. 146–200. This article is the English translation of Ref. 3.
 - [5] K. Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie," Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. **1916**, 189–196 (1916). Reprinted in translation as "On the gravitational field of a mass point according to Einstein's theory," arXiv: physics/9905030v1 [physics.hist-ph].
 - [6] J. Droste, "Het veld van een enkel centrum in Einstein's theorie der zwaartekracht, en de beweging van een stoffelijk punt in dat veld," Versl. Akad. Amst. **25**, 163–180 (1916-1917). Reprinted in translation as "The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field," Proc. K. Ned. Akad. Wetensch. **19** (1), 197–215 (1917), <adsabs.harvard.edu/abs/1917KNAB...19...197D>.
 - [7] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics* (Addison Wesley, San Francisco, 2002), 3rd ed., pp. 312–314 and p. 332 (Exercise 26).
 - [8] J. V. José and E. J. Saletan, *Classical Dynamics: A Contemporary Approach* (Cambridge University Press, Cambridge, 1998), pp. 209–212 and 276–277 (Problem 11).

- [9] P. C. Peters, “Comment on ‘Mercury’s precession according to special relativity’ [Am. J. Phys. **54**, 245 (1986)],” Am. J. Phys. **55** (8), 757 (1987).
- [10] T. Phipps, Jr., “Response to ‘Comment on “Mercury’s precession according to special relativity,”” [Am. J. Phys. **55**, 757 (1987)],” Am. J. Phys. **55** (8), 758 (1987).
- [11] T. Phipps, Jr., “Mercury’s precession according to special relativity,” Am. J. Phys. **54** (3), 245 (1986).
- [12] Reference 7, pp. 536–539.
- [13] S. T. Thornton and J. B. Marion, *Classical Dynamics of Particles and Systems* (Thomson Brooks/Cole, Belmont, CA, 2004), pp. 292–293 and 312–316.
- [14] V. D. Barger and M. G. Olsson, *Classical Mechanics: A Modern Perspective* (McGraw-Hill, Inc., New York, 1995), pp. 306–309.
- [15] G. R. Fowles, *Analytical Mechanics* (Holt, Rinehart and Winston, New York, 1977), pp. 161–164.
- [16] L. N. Hand and J. D. Finch, *Analytical Mechanics* (Cambridge University Press, Cambridge, 1998), pp. 420–422 (Problems 18 and 19).
- [17] Reference 13, pp. 578–579.
- [18] Reference 14, p. 366 (Problem 10-12).
- [19] J. M. Potgieter, “Derivation of the equations of Lagrange for a relativistic particle,” Am. J. Phys. **51** (1), 77 (1983).
- [20] E. A. Desloge and E. Eriksen, “Lagrange’s equations of motion for a relativistic particle,” Am. J. Phys. **53** (1), 83–84 (1985).
- [21] Y.-S. Huang and C.-L. Lin, “A systematic method to determine the Lagrangian directly from the equations of motion,” Am. J. Phys. **70** (7), 741–743 (2002).
- [22] S. Sonogo and M. Pin, “Deriving relativistic momentum and energy,” Eur. J. Phys. **26**, 33–45 (2005).
- [23] Reference 7, p. 87.
- [24] Reference 8, p. 84.
- [25] Reference 13, p. 292.
- [26] Reference 15, p. 148.
- [27] P. Hamill, *Intermediate Dynamics* (Jones and Bartlett Publishers, LLC, Sudbury, MA, 2010), p. 330.
- [28] Reference 7, pp. 92–96.
- [29] Reference 8, pp. 84–86.
- [30] Reference 13, pp. 300–304.
- [31] Reference 15, pp. 137–157.

- [32] P. Smith and R. C. Smith, *Mechanics* (John Wiley & Sons, Chichester, 1990), pp. 195–202.
- [33] Reference 27, pp. 307–343.
- [34] T. J. Lemmon and A. R. Mondragon, “Alternative derivation of the relativistic contribution to perihelion precession,” *Am. J. Phys.* **77** (10), 890–893 (2009); arXiv:0906.1221v2 [astro-ph.EP].
- [35] D. H. Frisch, “Simple aspects of post-Newtonian gravitation,” *Am. J. Phys.* **58** (4), 332–337 (1990).
- [36] S. M. Carroll, *An Introduction to General Relativity: Spacetime and Geometry* (Addison-Wesley, San Francisco, 2004), pp. 205–216.
- [37] J. Plebański and A. Kasiński, *An Introduction to General Relativity and Cosmology* (Cambridge University Press, Cambridge, 2006), pp. 176–182.
- [38] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), pp. 1110–1115.
- [39] S. Weinberg, *Gravitation and Cosmology* (John Wiley & Sons, New York, 1972), pp. 194–201.
- [40] W. Rindler, *Relativity: Special, General, and Cosmological* (Oxford University Press, New York, 2001), pp. 238–245.
- [41] H. Ohanian and R. Ruffini, *Gravitation and Spacetime* (W. W. Norton & Company, New York, 1994), pp. 401–408.
- [42] R. D’Inverno, *Introducing Einstein’s Relativity* (Oxford University Press, New York, 1995), pp. 195–198.
- [43] M. P. Hobson, G. P. Efstathiou, and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, 2006), pp. 205–216, 230–233.
- [44] K. Doggett, “Comment on ‘Precession of the perihelion of Mercury,’ by Daniel R. Stump [*Am. J. Phys.* **56**, 1097–1098 (1988)],” *Am. J. Phys.* **59** (9), 851–852 (1991).
- [45] D. R. Stump, “Precession of the perihelion of Mercury,” *Am. J. Phys.* **56** (12), 1097–1098 (1988).
- [46] G. Ovanessian, “Derivation of relativistic corrections to bounded and unbounded motion in a weak gravitational field by integrating the equations of motion,” *Am. J. Phys.* **71** (9), 912–916 (2003).
- [47] A. M. Nobili and I. W. Roxburgh, “Simulation of general relativistic corrections in long term numerical integrations of planetary orbits,” in *Relativity in Celestial Mechanics and Astrometry: High Precision Dynamical Theories and Observational Verifications*, edited by J. Kovalevsky and V. A. Brumberg (IAU, Dordrecht, 1986), pp. 105–110, <adsabs.harvard.edu/abs/1986IAUS...114..105N>.
- [48] C. Magnan, “Complete calculations of the perihelion precession of Mercury and the deflection of light

- by the Sun in general relativity,” unpublished, arXiv:0712.3709v1.
- [49] M. M. D’Eliseo, “The first-order orbital equation,” *Am. J. Phys.* **75** (4), 352–355 (2007).
 - [50] D. R. Brill and D. Goel, “Light bending and perihelion precession: A unified approach,” *Am. J. Phys.* **67** (4), 316–319 (1999).
 - [51] B. Dean, “Phase-plane analysis of perihelion precession and Schwarzschild orbital dynamics,” *Am. J. Phys.* **67** (1), 78–86 (1999).
 - [52] N. Ashby, “Planetary perturbation equations based on relativistic Keplerian motion,” in *Relativity in Celestial Mechanics and Astrometry: High Precision Dynamical Theories and Observational Verifications*, edited by J. Kovalevsky and V. A. Brumberg (IAU, Dordrecht, 1986), pp. 41–52, <adsabs.harvard.edu/abs/1986IAUS...114...41A>.
 - [53] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984), pp. 136–143.
 - [54] J. B. Hartle, *Gravity: An Introduction to Einstein’s General Relativity* (Addison-Wesley, San Francisco, 2003), pp. 191–204.
 - [55] A. Schild, “Equivalence principle and red-shift measurements,” *Am. J. Phys.* **28** (9), 778–780 (1960).
 - [56] Reference 54, pp. 230–232.
 - [57] M. G. Stewart, “Precession of the perihelion of Mercury’s orbit,” *Am. J. Phys.* **73** (8), 730–734 (2005).
 - [58] D. Brouwer and G. M. Clemence, “Orbits and masses of planets and satellites,” in *The Solar System: Planets and Satellites*, edited by G. P. Kuiper and B. M. Middlehurst (University of Chicago Press, Chicago, 1961), Vol. III, pp. 31–94.
 - [59] C. Sigismondi, “Astrometry and relativity,” *Nuovo Cim. B* **120**, 1169–1180 (2005).
 - [60] Reference 38, pp. 660–662, Box 25.6.
 - [61] F. Y.-H. Wang, “Relativistic orbits with computer algebra,” *Am. J. Phys.* **72** (8), 1040–1044 (2004).
 - [62] A. Harvey, “Newtonian limit for a geodesic in a Schwarzschild field,” *Am. J. Phys.* **46** (9), 928–929 (1978).
 - [63] W. Rindler, *Special Relativity* (Oliver and Boyd, London, 1960), pp. 79–103.
 - [64] J. L. Synge, *Relativity: The Special Theory* (North-Holland Publishing Company, Amsterdam, 1958), pp. 394–399.

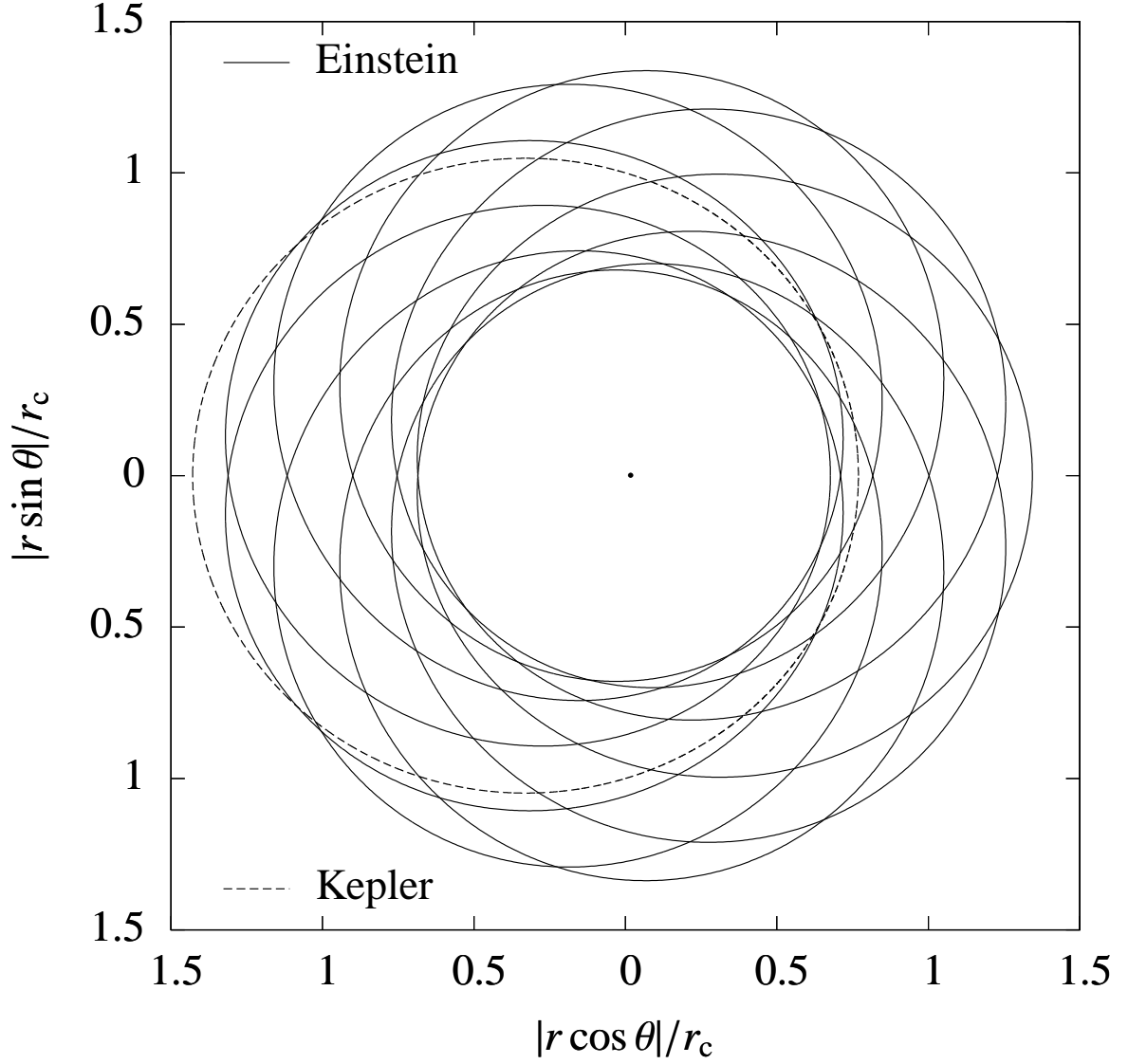


FIG. 1: Relativistic orbit in a Keplerian limit (solid line), as described by Eq. (25), compared to a corresponding Keplerian orbit (dashed line) [Eq. (11)]. The precession of perihelion is one orbital characteristic due to special relativity and is illustrated here for $0 \leq \theta \leq 20\pi$. This characteristic is exaggerated by both the choice of eccentricity ($e = 0.3$) and relativistic correction parameter ($\epsilon = 0.2$) for purposes of illustration. Precession is present for smaller (non-zero) reasonably chosen values of e and ϵ as well. (The same value of e is chosen for both curves.)

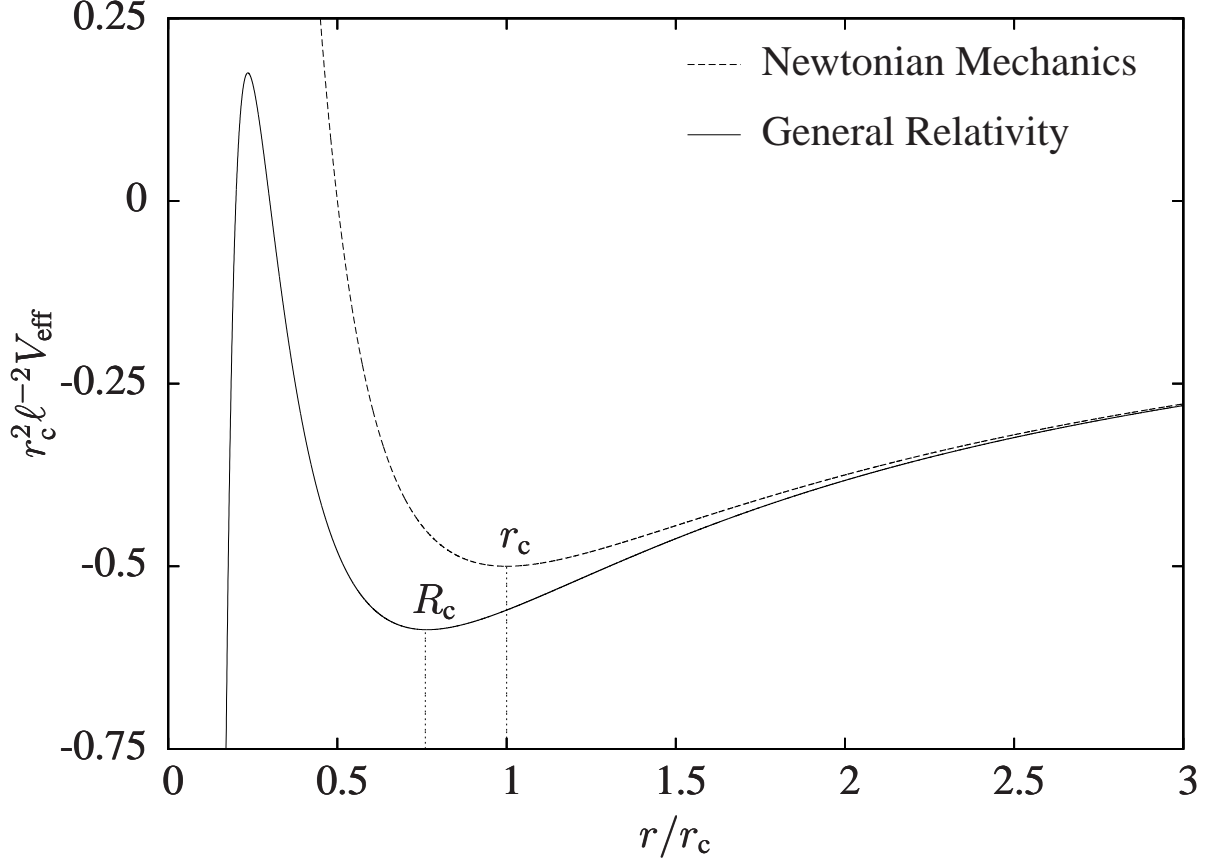


FIG. 2: Effective potential commonly defined in the Newtonian limit to General Relativity (solid line) [Eq. (31)], compared to that derived from Newtonian mechanics (dashed line). The vertical dotted lines identify the radii of circular orbits, R_c and r_c , as calculated using general relativity and Newtonian mechanics, respectively. General relativity predicts [Eq. (32)] a smaller radius of the circular orbit than that predicted by Newtonian mechanics. The value $\epsilon = 0.06$ is chosen for purposes of illustration.